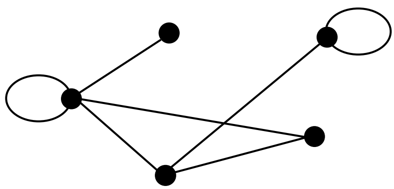
1. DEFINITIONS AND SIMPLE PROPERTIES

§1.1. Graphs

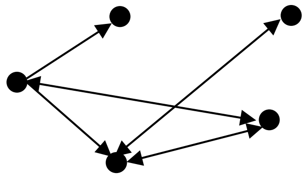
In mathematics the word ‘graph’ is used to describe a picture of a function $y = f(x)$ against two axes. But there’s another sort of graph – one that is studied in Graph Theory. A graph is a collection of objects, called **vertices**, and connections between vertices, called **edges**.



I don’t, however, allow multiple edges between the same pair of vertices, though some references do allow this.

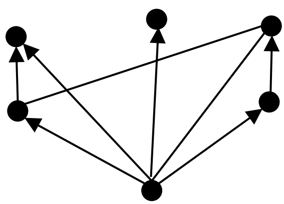
I allow **loops**, that is, edges from a vertex to itself, though in many applications our graphs will be **loop-free**, that is they’ll have no loops.

In general the edges of a graph may have a direction. If so, we represent the edges in the picture of the graph as arrows. Such a graph is called a **directed graph**. For example, friendship is often one way. Jack might consider Joe as a friend but Joe might not regard Jack as a friend.



A directed graph is nothing more than a relation on the set of vertices. A reflexive relation will be represented by a graph with a loop at each vertex. A symmetric relation will be represented by a directed graph in which each arrow goes in both directions. In such cases we can remove the arrow heads and we have an **undirected graph**.

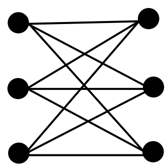
Example 1: The relation of divisibility on the set {1, 2, 3, 4, 5, 6} can be represented by the graph:



I have deliberately not labelled the vertices. See if you can work out which number is represented by each vertex.

Example 2: See if you can work out what well-known relation on {1, 2, 3, 4}
is represented by the directed graph on the right.

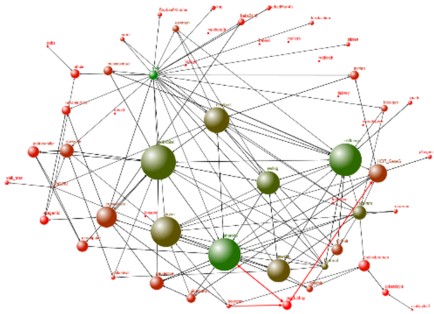
We often represent vertices by points and edges by lines and when we draw a graph on a surface it can tell us something about the topology of the surface. For example, we can prove that the following graph can't be drawn on a plane without some edges crossing. However, it can be so drawn on a doughnut without the edges crossing.



But very often graphs have no geometric or topological significance. For example the vertices might represent people and the edges might connect pairs of mutual friends.

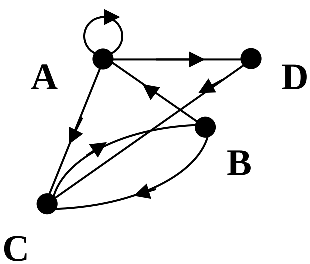
§1.2. Basic Definitions for Graphs

A **directed graph** is a relation on a set X, of elements together with a relation on X. Usually I shall consider graphs where X is finite. The reason for having this alternative terminology as that we think of a graph pictorially. The elements of X are represented by dots, called **vertices**, and the ordered pairs, called **edges**, that make up the relation are represented by arrows connecting certain dots to others. If E is the directed edge $V_1 \rightarrow V_2$ we call V_1 the **start** of E and V_2 is called the **end** of E. A **loop** is an edge (V, V) from a vertex to itself.



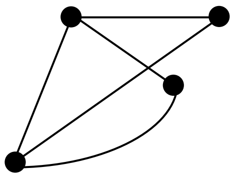
A **graph** (or undirected graph) is a symmetric relation on a set X. Here we replace the arrows going in both directions by lines. If E is an edge connecting vertices A and B we say that A and B are **adjacent**, and we denote the edge by **AB** (which, of course, is the same as BA).

Example 3: The following is a graph on the set {A, B, C, D}:
{(A, A), (A, C), (A, D), (B, A), (B, C), (C, B), (D, C)}. This can be represented by the following diagram. There are 4 vertices and 7 edges.



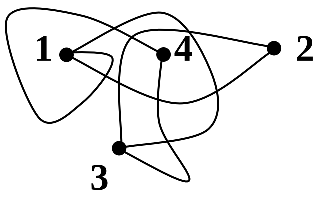
The edges of this graph can be listed very concisely as AA, AC, AD, BA, BC, DC.

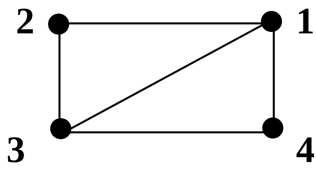
Example 4: The following is an undirected graph with no loops.



A graph is a combinatorial structure where the only consideration is which vertices are adjacent to which. When we draw a graph the positions of the points representing the vertices are arbitrary. So are the routes of the edges. The edges needn't be straight – they're allowed to cross over other edges, and they could even wind around in more complicated ways. However we usually draw a graph in such a way that it gives as simple a picture as possible.

Example 5: The graph in example 4 could be redrawn as in the diagram on the left, but would look much better when drawn as the one on the right.





A **walk** in a graph is a sequence of edges E_1, E_2, … E_m where, for i = 1, …, m – 1, the end of E_i is the same vertex as the start of E_{i+1}. You can think of it as representing your movement around a maze. You can repeat edges, go around loops and even make U-turns where arrows go in both directions. A **closed walk** is one where the start of the first edge is the same as the end of the last. That is, you end up where you begin. For an undirected graph just remember that each edge goes in both directions. The **length** of a walk is the number of edges it contains. We consider a walk consisting of no edges as a walk of length zero.

A vertex u is **connected** to a vertex v if there is a walk from u to v.

The **adjacency matrix** of a graph on n vertices is the $n \times n$ matrix (a_{ij}) where $a_{ij} = 1$ if there is an edge $i \to j$ in the graph (the vertices being numbered 1 to n).

Theorem 1: If A is the adjacency matrix for a graph, the i-j component of A^m is the number of walks of length m from i to j.

Proof: We prove this by induction on m. Clearly the i-j component of A is the number of walks of length 1, that is edges, from i to j, so the theorem holds for $m = 1$. (Actually we could start the induction at $m = 0$ since the only walks of length 0 are those from a vertex to itself, which corresponds to the identity matrix.)

Suppose the theorem holds for m. We shall count the walks of length $m + 1$. Every such walk will consist of an edge (walk of length 1), followed by a walk of length m.

So the number of such walks is $\sum_{i=1}^{n} a_{ik} b_{kj}$ where a_{ik} is the i-k component of A and b_{kj} is the number of walks of length m from k to j.

By induction, b_{kj} is the j-k component of A^m.

But $\sum_{i=1}^{n} a_{ik} b_{kj}$ is the i-j component of $AA^m = A^{m+1}$.

And so the theorem holds for $m + 1$.

Therefore, by induction, it holds for all m. ✋☺

Corollary: In a graph with n vertices, vertex i is connected to vertex j if and only if the i-j component of:

$$I + A + \ldots + A^{n-1}$$

is positive.

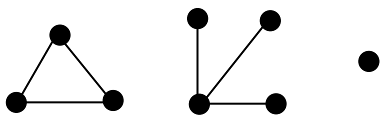
Proof: If there's a walk from i to j then there will be one of length at most $n - 1$.

A **path** in a graph is a walk where no edge is repeated. However vertices may be repeated.

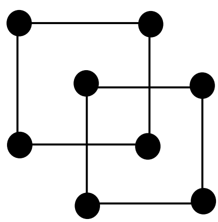
From now on, unless otherwise stated, I will assume that all graphs are undirected, finite and have no loops.

The relation of vertices being connected is, for an undirected graph, an equivalence relation. The equivalence classes are called the **components** of the graph. A graph is **connected** if it has only one component.

Example 6: The following graph is not connected. It has 3 components.



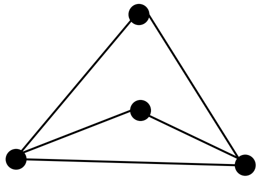
A badly drawn picture of a non-connected graph might make the graph appear connected, but note that the following graph has 2 components.

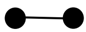


§1.3. Isomorphic Graphs

Two graphs X and Y are **isomorphic** if there's a 1-1 and onto map $F: X \to Y$ such that V_1 and V_2 are adjacent in X if and only if $F(V_1)$ and $F(V_2)$ are adjacent in Y. If G, H are isomorphic we write $G \cong H$.

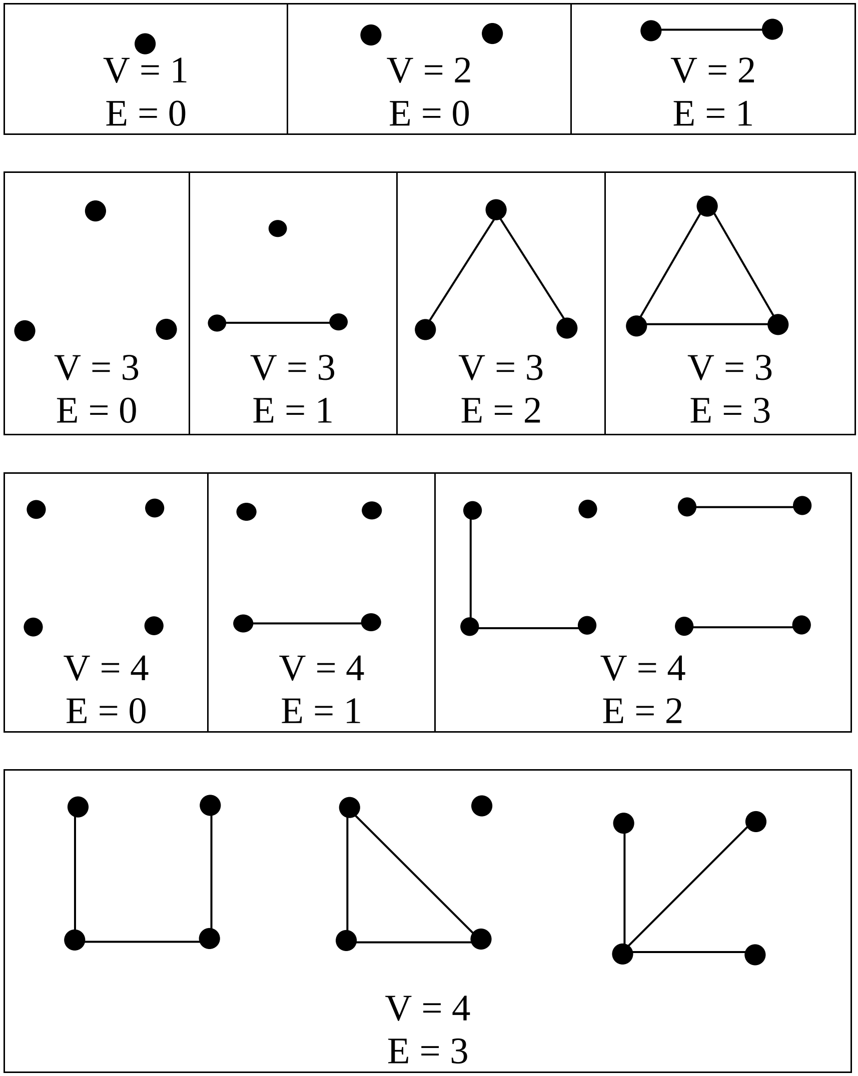
Example 7: The two graphs in Example 5 are isomorphic. And both are isomorphic to the following graph.

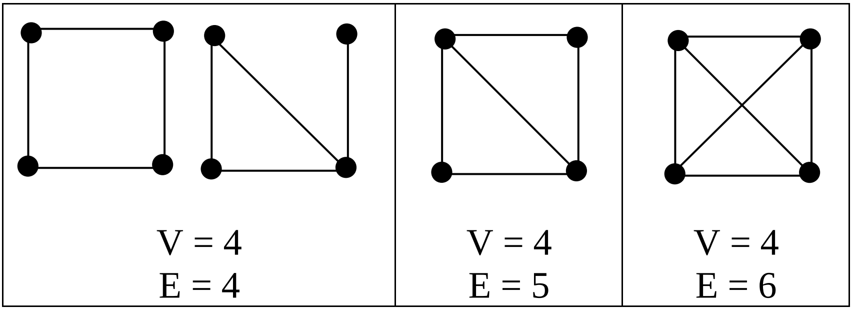


We often consider isomorphic graphs as being the same graph (unless they are related in some way). So while there are infinitely many undirected graphs with two vertices, there are only two isomorphism classes. Every graph on two vertices is isomorphic to either  or  so we say that there are only two (undirected) graphs with two vertices.

Clearly there are finitely many graphs with a given number of edges so we can provide a list of all graphs with up to a certain number of vertices. Here is a catalogue of all undirected graphs with 4 vertices or less. They have been systematically classified according to the number of vertices, V and edges, E.

Example 6: The following list contains all the graphs with 4 vertices or less. Every graph with up to 5 vertices is equivalent to exactly one of these. They've been systematically classified according to the number of vertices, V, and edges, E.

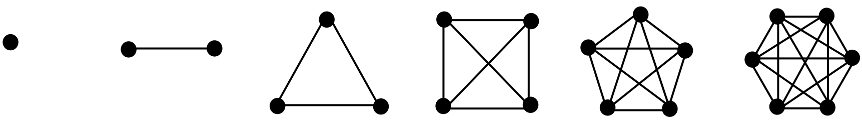

V = 1
E = 0
V = 2
E = 0
V = 2
E = 1
V = 3
E = 0
V = 3
E = 1
V = 3
E = 2
V = 3
E = 3
V = 4
E = 0
V = 4
E = 1
V = 4
E = 2
V = 4
E = 3



§1.4. Complete Graphs

The **complete graph** on *n* vertices, denoted by $\mathbf{K}_n$, is the graph where every vertex is adjacent to every other vertex. The number of edges in $\mathbf{K}_n$ is clearly the binomial coefficient $\binom{n}{2}$. The graph K_3 is called a **triangle**.

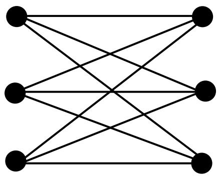
Example 7: The following are the complete graphs on 6 vertices or less.



$\mathbf{K}_1$ $\mathbf{K}_2$ $\mathbf{K}_3$ $\mathbf{K}_4$ $\mathbf{K}_5$ $\mathbf{K}_6$

Another important family of graphs consists of the graphs $\mathbf{K}_{m,n}$ for various values of m and n (they don't have a name, just a symbol). The graph $\mathbf{K}_{m,n}$ has $m + n$ vertices divided into two subsets, one of size m and the other of size n. Every vertex in one subset is adjacent to every vertex in the other, but there are no edges connecting two vertices within the same subset.

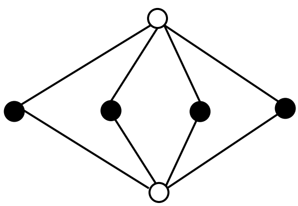
Example 8: The following is $\mathbf{K_{3,3}}$:



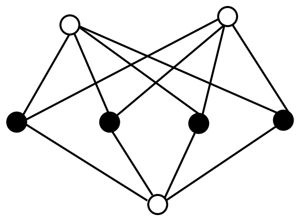
This graph was once featured in an Air New Zealand advertisement, where the 6 vertices consisted of the cities Brisbane, Sydney, Melbourne, Auckland, Wellington and Christchurch. The edges represented the trans-Tasman routes.

Example 8:

$\mathbf{K_{4,2}}$



$\mathbf{K_{4,3}}$

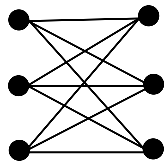


§1.5. Subgraphs

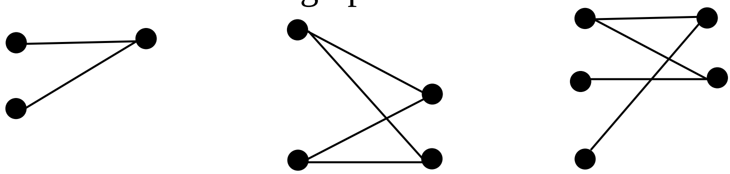
A **subgraph** of a graph G is a subset of G together with some or all of the edges that connect them.
Every graph is a subgraph of itself. The smallest subgraph of any graph is K_1, ● consisting of one vertex and no edges:

We don’t need to include all the edges of G that connect vertices in the subset. For example we could take all the vertices of G and none of its edges.

Example 7: $K_{3,3}$ is the following graph:



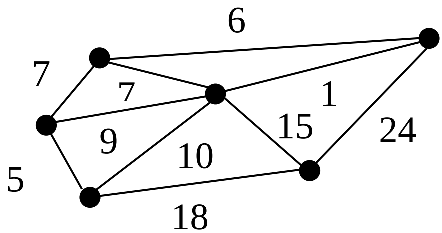
Here are some of its subgraphs:



Notice that $K_{3,3}$ has no subgraphs that are isomorphic to K_3. that is, it has no triangles.

§1.6. Weighted Graphs

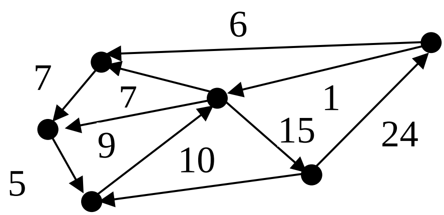
A graph (either directed or undirected) is called **weighted** if there is a number associated with each edge. A famous problem in computing science is the so-called Travelling Salesman Problem. Given a weighted graph, of all the paths that start at one vertex, visit every other vertex and return to the one where it started, find one for which the sum of the weights is least.



Note that the numbers in this example can't represent distances because there's a triangle with sides 6, 7 and 14. In a geometric triangle the length of each side can't exceed the sum of the other two sides. But the numbers could refer to the capacity of pipes in a plumbing network. Or the vertices could represent a group of friends and the numbers might represent the number of Facebook messages between each pair on a given day. Although we often draw graphs pictorially don't assume that they must always represent something geometric.

There's a path in the above graph that goes through each vertex, returning to the start, whose total length is only 69. Can you find it? Is this the best possible?

Weighting in graphs can apply equally well to directed graphs.
For example if some of the roads in the above example were one-way we'd have a weighted directed graph. What now is the shortest path that visits each vertex and returns to the start?



§1.7. The Complement of a Graph

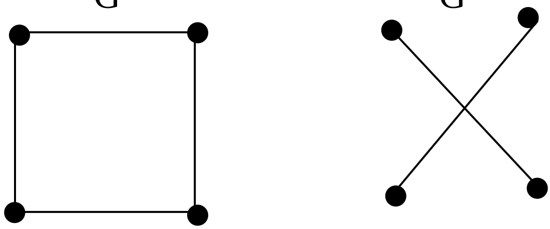
The **complement** of a graph is the graph G on the same set of vertices where two vertices are adjacent in G if and only if they are not adjacent in the complement.

The complement of G is denoted by $\overline{\mathbf{G}}$ and clearly $\overline{\overline{G}}$ is isomorphic to G.

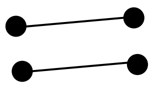
If G is on *m* vertices and has *n* edges then $\overline{G}$ is on the same *m* vertices but with $\binom{m}{2} - n$ edges.

Example 8:

G $\qquad\qquad$ $\overline{G}$

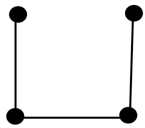
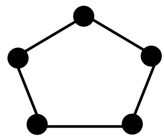


$\overline{G}$ is isomorphic to:



A graph G is **self complementary** if $G \cong \overline{G}$.

Example 9: The following are all the self-complementary graphs with up to 5 vertices.

		
V = 1	V = 4	V = 5
E = 0	E = 3	E = 5

Theorem 2: If G is a self-complementary graph with V vertices then V ≡ 0 or 1 modulo 4.

Proof: The maximum possible number of edges in a graph G with V vertices is ½ V(V – 1). If G is self complementary this must be even so that G and its complement each have exactly half this number of edges.

If V = 4*k* then G must have *k*(4*k* – 1) edges.

If V = 4*k* + 1 G must have (4*k* + 1)k edges.

If V = 4*k* + 2, G must have ½ (2*k* + 1)(4*k* + 1) edges, which is impossible.

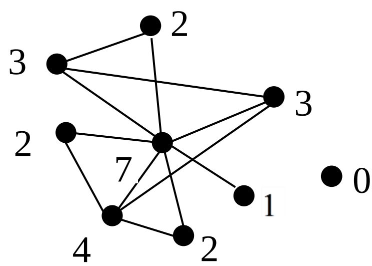
If V = 4*k* + 3, G must have ½ (4*k* + 3)(2*k* + 1) edges, which is impossible. ✋☺

So the next largest self-complementary graph will have 8 vertices and 14 edges.

§1.8. The Degree of a Vertex

The **degree** of a vertex in an undirected graph is the number of vertices adjacent to it (in other words it is the number of edges that have that vertex as one of its endpoints). We denote the degree of vertex V by **deg(V)**.

Example 10: In the following graph the degrees of the vertices are indicated.



Notice that there are 9 vertices, 12 edges and the sum of the degrees is 24. Is there a connection between these numbers?

The **degree** list for a graph G is a list of the degrees of the vertices in ascending order. It is denoted by $\Delta(G)$.

Example 11: If G is the graph in Example 10, $\Delta(G) = (0, 1, 2, 2, 2, 3, 3, 4, 7)$.

An associated concept is the **degree equation** for a graph. This is the equation:

$$2E = d_1 + d_2 + \ldots + d_V$$

where $d_1 \geq d_2 \geq \ldots \geq d_V$ are the degrees in descending order.

This reflects the following theorem.

Theorem 3: If G is a graph with V vertices and E edges, each degree is at most V – 1 and the sum of the degrees is 2E.
Proof: The first part of this theorem is obvious. For the second, each edge contributes 2 to the sum of the degrees, one for each end. Hence the sum of the degrees is twice the number of edges. ✋☺

For a large graph it is often more convenient to use the compact version of the degree equation.

$$2E = d_1 * n_1 + d_2 * n_2 + \ldots + d_k * n_k,$$

Where the d_i's are the distinct degrees and n_i is the number of vertices with degree d_i. If $n_i = 1$ we omit the '∗1' In this case the order of the terms can vary.

Example 12: The graph in Example 10 has degree sequence:

$$\Delta(G) = (0, 1, 2, 2, 2, 3, 3, 4, 7).$$

It's degree equation is:

$$24 = 7 + 4 + 3 + 3 + 2 + 2 + 2 + 1 + 0,$$

or more compactly,

$$24 = 0 + 1 + 2*3 + 3*2 + 4 + 7.$$

Remember that with $d*n$, the first number is the degree and the second is the multiplicity.

Example 13: Prove that there is no graph G where

$$\Delta(G) = (1, 2, 2, 2, 3, 4, 4, 5).$$

Solution: The sum of the degrees is 23, which is odd. Hence such a graph is impossible.

Example 14: Prove that there is no graph G where

$$\Delta(G) = (1, 2, 2, 2, 3, 4, 4, 8).$$

Solution: Since there are 8 vertices the degree of each vertex must be at most 7.

Example 15: Prove that there is no graph G where

$$\Delta(G) = (1, 1, 2, 3, 3, 3, 6, 7).$$

Solution: Since there are 8 vertices the vertex of degree 7 must be at most adjacent to every other vertex. Removing this vertex, and the 7 edges at that vertex we would get a subgraph with 7 vertices and degrees 0, 0, 1, 2, 2, 2, 5. Removing the vertices of degree 0 we would get a graph with degrees 1, 2, 2, 2, 5. This is impossible because there are only 4 vertices for the vertex of degree 5 to be adjacent to.

Example 16: Draw a graph G where

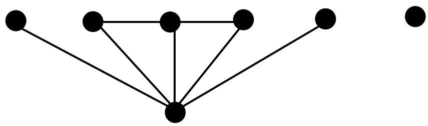
$$\Delta(G) = (1, 2, 2, 3, 3, 4, 6, 7).$$

Solution: There are 8 vertices and so the vertex of degree 7 must be adjacent to every other vertex. Removing this vertex, and the 7 edges at that vertex we would get a subgraph H with 7 vertices and degrees 0, 1, 1, 2, 2, 3, 5. Removing the vertex of degree 0 we would get a graph with degrees 1, 1, 2, 2, 3, 5. The vertex of degree 5 would be adjacent to every other vertex in this subgraph. Hence, as before, we reach a graph K with degrees 0, 0, 1, 1, 2. Clearly this graph must be:

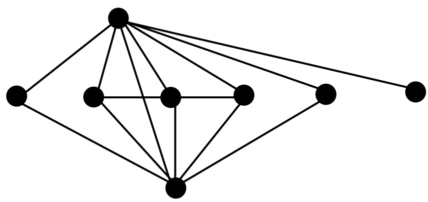
K

Working backwards we see that H and G must be as follows.

H



G



By examining the way the solution was obtained we can see that all graphs whose 8 vertices have degrees 1, 2, 2, 3, 3, 4, 6, 7 must be equivalent.

If a vertex V in a graph G with V vertices has degree *d* then it has degree (V – 1) – *d* in the complement, because there are V –1 other vertices and those that are adjacent in G are not adjacent in the complement.

So if G is self-complementary Δ(G) must be unchanged after transforming them by $d \to (V - 1) - d$, and reversing the order.

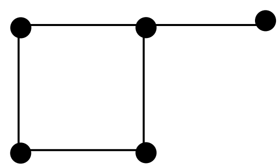
Example 17: If G, H, K are the three graphs in Example 16 then Δ(G) = (1, 1, 2, 2), Δ(H) = (1, 1, 2, 3, 3) and

Δ(K) = (2, 2, 2, 2, 2).

For G, V – 1 = 3 and transforming by $d \to (V - 1) - d$, Δ(G) becomes (2, 2, 1, 1) which on reversing becomes (1, 1, 2, 2), that is Δ(G). For H, K, V – 1 = 4

The fact that Δ(G) is unchanged by transforming by $d \to (V - 1) - d$ and reversing does not guarantee that G is self-complementary.

Example 18: Let G be the graph:



then Δ(G) = (1, 2, 2, 2, 3) is unchanged by the transformation $d \to 4 - d$ and reversing, yet G is not self-complementary.

§1.9. Centres and Radii

Suppose G is a connected graph. The **distance** between two vertices, u and v in a connected graph is **d(u, v)**, the length of the shortest path between them.

The **diameter**, **d(G)**, of G is the maximum distance between any two vertices.

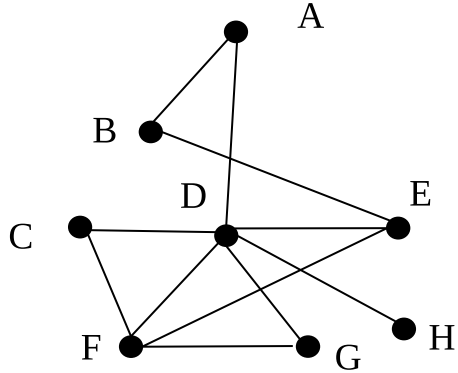
The **eccentricity**, **e(v)**, of a vertex v is the maximum distance d(u, v) for all vertices u.

The **radius**, **ρ(G)**, of a connected graph G is the minimum eccentricity of any vertex.

A **central** vertex is one where e(v) = ρ(G). The **centre** of G, **Z(G)**, is the set of central vertices.

distance	d(u, v)	length of shortest path
diameter	d(G)	max d(u, v) for u, v
eccentricity	e(v)	max d(u, v) for u
radius	ρ(G)	min e(v) for v
centre	Z(G)	{v | e(v) = ρ(G)}

Example 19: Consider the following graph Γ.



The distances are as follows:

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
A	0	1	2	2	1	2	2	2
B	1	0	1	3	2	3	2	3
C	2	1	0	2	1	2	1	2
D	2	3	2	0	1	2	1	2
E	1	2	1	1	0	1	1	1
F	2	3	2	2	1	0	2	2
G	2	2	1	1	1	2	0	1
H	2	3	2	2	1	2	1	0

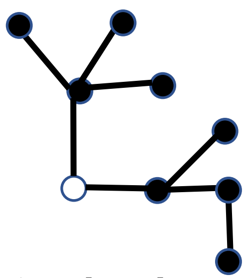
$d(\Gamma) = 3$.

The eccentricities are as follows:

A	B	C	D	E	F	G	H
2	3	2	3	2	3	2	3

$\rho(\Gamma) = 2$ and $Z(\Gamma) = \{A, C, E, G\}$.

Example 20: Let Γ be the following tree.



$d(\Gamma) = 5$, $\rho(\Gamma) = 3$. Z(G) is the white vertex.